Mathematics of Quantum Mechanics on Thin Structures

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Lecture 3

Possible, Impossible, and Extremal Spectra

Can one hear the shape of a drum? Of a potential?

Sometimes, yes.

But first... How can you determine where the spectrum is discrete and where continuous?

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- In QM, discrete eigenvalues are *bound states*,like the ones that are localized and last indefinitely as atomic energy levels
- In QM, the continuous spectrum is associated with dynamic processes, such as scattering and electric current.

How can you determine where the spectrum is discrete and where continuous?

- A theorem of Weyl states that A and B have the same *essential spectrum* if A-B is compact.
  - in practice, we more often establish that $(A + z)^{-1} - (B + z)^{-1}$ is compact. This often entails showing that a certain integral is finite. Example: Any V(**x**) bounded and compactly supported is "relatively compact" with respect to $-\nabla^2$.

How can you determine where the spectrum is discrete and where continuous?

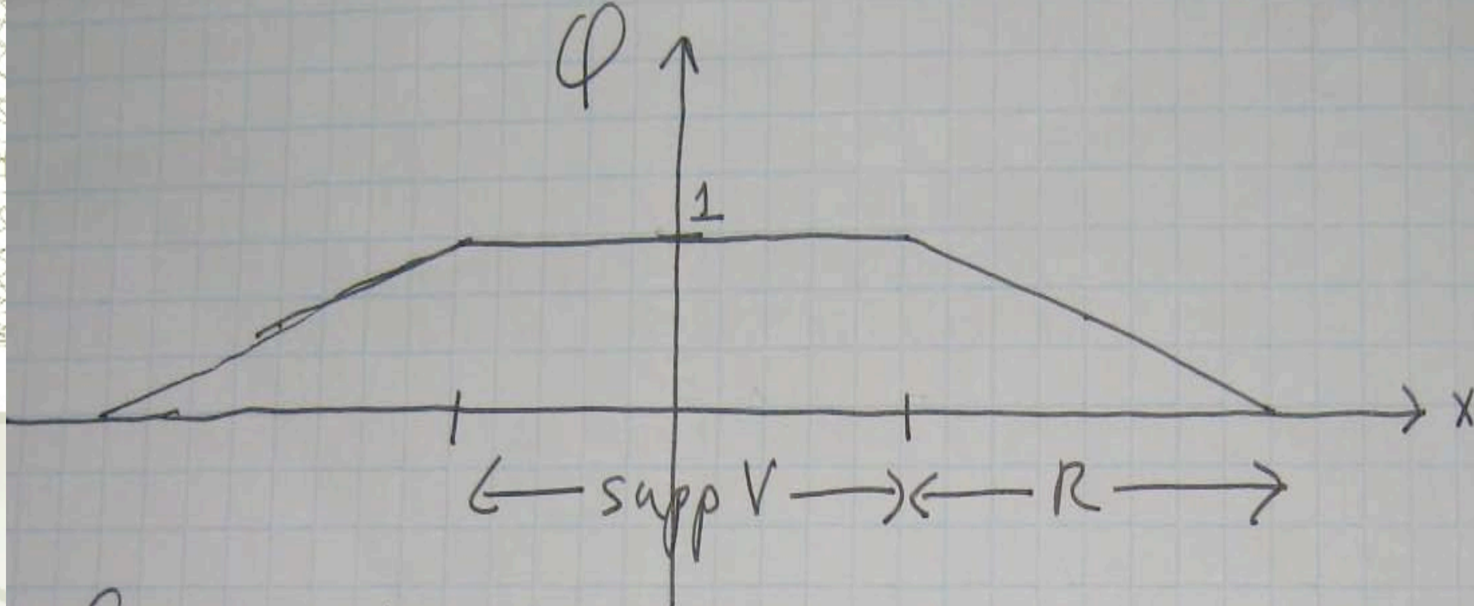
- Suppose $V(x)$ bounded and compactly supported, with negative integral on R. Then $-d^2/dx^2 + V$ has a negative eigenvalue. In QM this is a *bound state*.

Proof

1. Because V is relatively compact, the essential spectrum ≥ 0.
2. Cook up a trial function φ which = 1 on supp V, but goes to 0 over a distance R from supp V. We shall show that the Rayleigh quotient can be made negative. Therefore there is spectrum below 0, and since it is not essential, it consists of one or more eigenvalues.

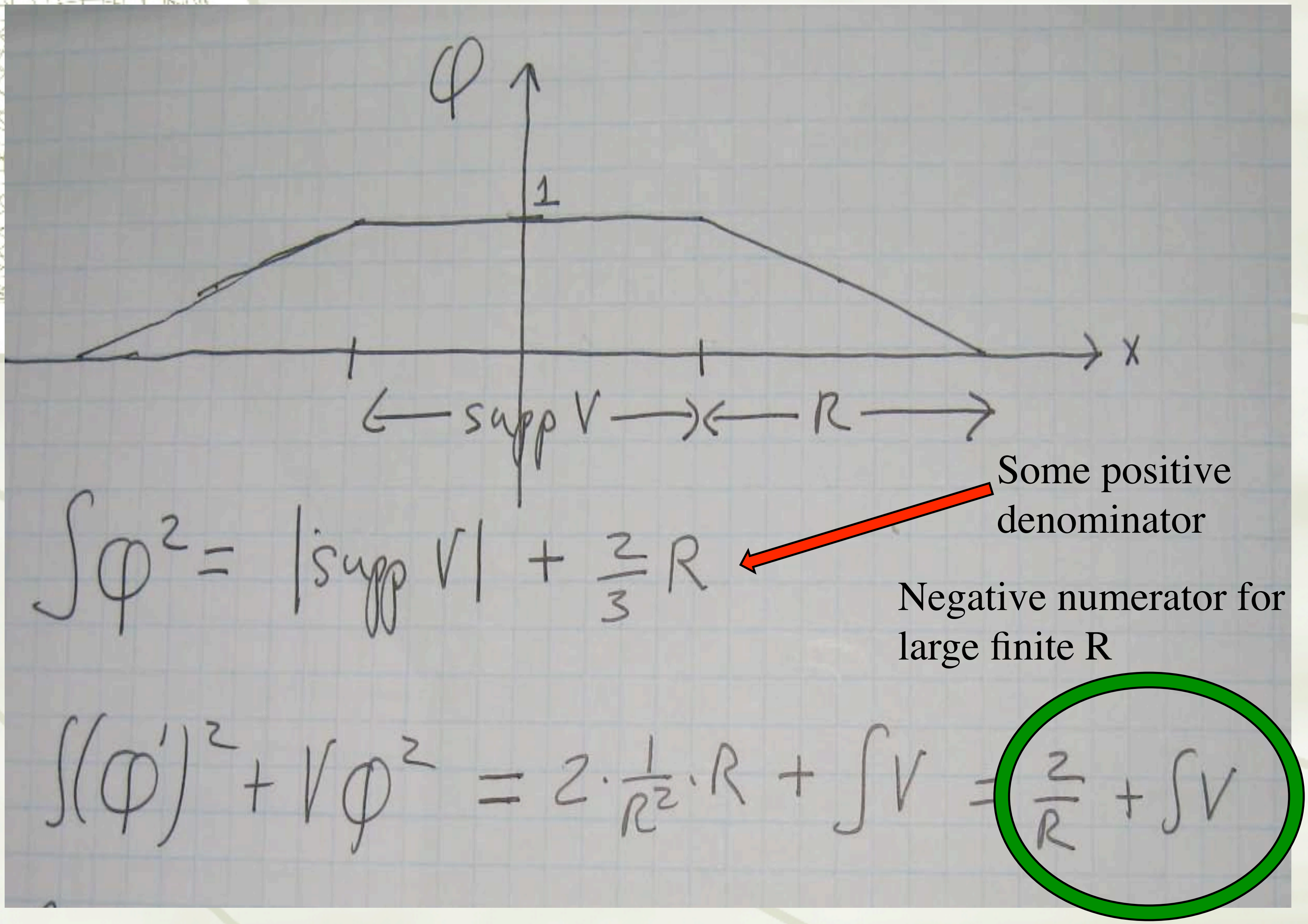
Rayleigh quotient

$$R(\zeta) = \frac{\int_\Omega (|\nabla\zeta|^2 + V(\mathbf{x})|\zeta|^2)dV}{\int_\Omega |\zeta|^2 dV}$$



$$\int \varphi^2 = |\operatorname{supp} V| + \frac{2}{3}R$$

$$\int (\varphi')^2 + V\varphi^2 = 2 \cdot \frac{1}{R^2} \cdot R + \int V = \frac{2}{R} + \int V$$



Some positive denominator

Negative numerator for large finite R

Application to nanotechnology

- (Duclos-Exner) Suppose we make a *quantum wire* by building a small tube of radius $\delta \to 0$ around a smooth curve. Unless the curve is absolutely straight, the effective potential $-\kappa^2/4$ is ≤ 0 and strictly negative on a set of positive measure. Conclusion....

Do large eigenvalues exhibit any regular patterns?

Do large eigenvalues exhibit any regular patterns?

- Yes. According to a theorem of Hermann Weyl, if N(z) denotes the number of eigenvalues of $-\nabla^2 \leq z$, then

$$N(z) \sim \frac{C_d |\Omega| z^{d/2}}{(2\pi)^d}$$

$$C_d = \pi^{d/2} / \Gamma(1 + d/2)$$

(True whether Dirichlet problem on domain or manifold.)

Do large eigenvalues exhibit any regular patterns?

- Yes. According to a theorem of Hermann Weyl, if N(z) denotes the number of eigenvalues of $-\nabla^2 \leq z$, then

$$\lambda_n \sim 4\pi \left(\frac{\Gamma(1 + d/2) n}{|\Omega|} \right)^{2/d}$$

Do large eigenvalues exhibit any regular patterns?

- For a Schrödinger operator $-\nabla^2 + V(\mathbf{x})$,

$$\lambda_n \sim 4\pi \left(\frac{\Gamma(1 + d/2)n}{|\Omega|} \right)^{2/d} + \frac{V_{\text{ave}}}{2}$$

  - The high eigenvalues tell us the size of the domain and the average of the potential

Are different eigenvalues more or less independent?

- Ashbaugh-Benguria: λ_2/λ_1 is maximized uniquely by the ball.
- Pólya conjecture: The Weyl asymptotic formula for $N(\lambda)$ is an upper bound for each finite λ. (I.e., the estimate $\lambda_k \sim C(k/|\Omega|)^{2/d}$ is a lower bound.)
- Berezin-Li-Yau: The “integrated” version of Pólya for sums of eigenvalues is true.

The top and the bottom of the spectrum are connected

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On a closed manifold, the lowest eigenvalue of $-\nabla^2$ is trivial, since we know its spectrum is nonnegative, and we notice that $-\nabla^2 1 = 0$.

The top and the bottom of the spectrum are connected

Consider $H = -\nabla^2 + V(x)$. If we fix the integral of V, then the lowest eigenvalue λ_1 is maximized when V is constant. (Original 1-D theorem of this type was due to Ambarzumian, 1929.)

The top and the bottom of the spectrum are connected

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Proof

Recall the Rayleigh-Ritz inequality,

$$\lambda_1 \langle\zeta, \zeta\rangle \leq \langle\zeta, (-\nabla^2 + V)\, \zeta\rangle$$

And choose $\zeta = 1$. We see that $\lambda_1 \leq V_{ave}$. $V = cst$ is a case of equality. To see that it is the unique such case, suppose that

$$-\nabla^2 + V - V_{ave}$$

is a positive operator and use the spectral theorem to define $B \geq 0$ such that $B^2 = -\nabla^2 + V - V_{ave}$.

Proof

Calculate

$$\|B\ 1\|^2 = \langle B\ 1, B\ 1\rangle = \langle 1, (-\nabla^2 + V - V_{ave})1\rangle = 0.$$

Therefore $B\ 1 = 0$ so $0 = B^2\ 1 = V(x) - V_{ave}$. QED.

What constraints does non-commutation place on the spectrum?

- $[A,B] := AB - BA$
- Heisenberg: $xp - px = i$

On a (hyper) surface, what object is most like the good old flat Laplacian?

- **Answer #1 (Beltrami's answer): Consider only tangential variations.**

- **At a fixed point, orient Cartesian x_0 with the normal, then calculate**

$$\sum_{j=1}^{d} \frac{\partial^2}{\partial {x_j}^2}$$

Difficulty:

- **The Laplace-Beltrami operator is an intrinsic object, and as such is unaware that the surface is immersed!**

- **Answer #2 (The nanoanswer):**

$$- \Delta_{LB} + q$$

- **Perform a singular limit and renormalization to attain the surface as the limit of a thin domain.**

Difficulty:

- **Tied to a particular physical model - other effective potentials arise from other physical models or limits.**

Some other answers

- In other physical situations, such as reaction-diffusion, $q(x)$ may be other quadratic expressions in the curvature, usually $q(x) \leq 0$.
- The conformal answer: $q(x)$ is a multiple of the scalar curvature.

Heisenberg's Answer

(if he had thought about it)

$$q(\mathbf{x}) = +\frac{1}{4}\left(\sum_j \kappa_j\right)^2$$

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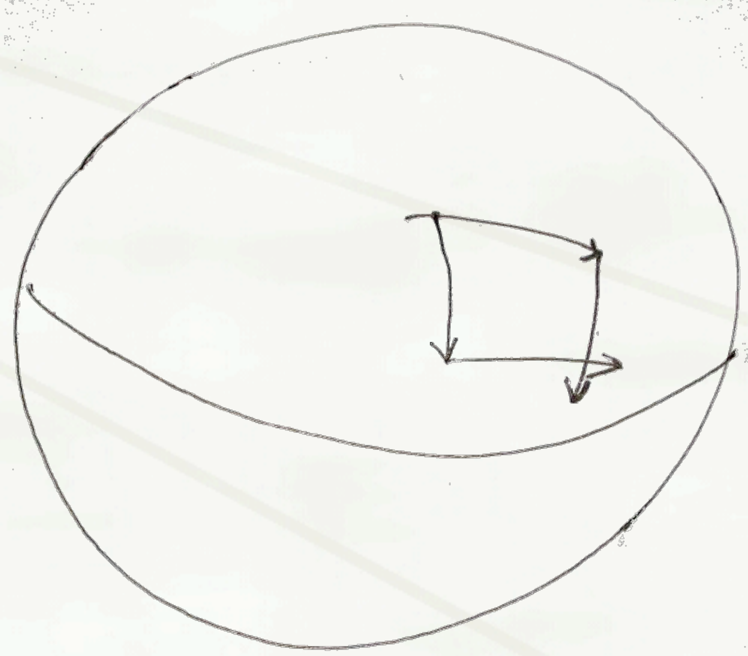
Note: $q(\mathbf{x}) \geq 0$!

Commutators: [A,B] := AB-BA

3. Curvature is the effect that motions do not commute:



Commutators: [A,B] := AB-BA

- More formally (from, e.g., Chavel, *Riemannian Geometry, A Modern Introduction*: Given vector fields X,Y,Z and a connection ∇, the curvature tensor is given by:

$$R(X,Y) = [\nabla_Y , \nabla_X] - \nabla_{[Y,X]}$$

Commutators: [A,B] := AB-BA

3a. The equations of space curves are commutators:

$$\frac{d\mathbf{x}}{ds} = \mathbf{t}$$

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n}$$

Commutators: [A,B] := AB-BA

3a. The equations of space curves are commutators:

$$\left[\frac{d}{ds}, \mathbf{x}\right] \varphi = \mathbf{t}\varphi$$

$$\left[\frac{d}{ds}, \mathbf{t}\right] \varphi = \kappa \mathbf{n}\varphi$$

Note: curvature is defined by a second commutator

The fundamental eigenvalue gap

$\Gamma := \lambda_2 - \lambda_1$

- In quantum mechanics, an excitation energy
- In "spectral geometry" a geometric quantity

  small gaps indicate decoupling (dumbbells)

  (Cheeger, Yang-Yau, etc.)

  large gaps indicate convexity/isoperimetric

  (Ashbaugh-Benguria)

Commutators and gaps

Elementary gap formula:

$$\langle u_j, [H,G]u_k \rangle = (\lambda_j - \lambda_k) \langle u_j, G u_k \rangle . \qquad (1.2)$$

Since $[H,G]u_k = (H - \lambda_k) G u_k$,

$$\|[H,G]u_k\|^2 = \left\langle G u_k, (H - \lambda_k)^2 G u_k \right\rangle , \qquad (1.3)$$

and more generally

$$\langle [H,G]u_j, [H,G]u_k \rangle = \langle G u_j, (H - \lambda_j)(H - \lambda_k) G u_k \rangle . \qquad (1.4)$$

Second commutator formula:

$$\langle u_j \mid [G,[H,G]]\, u_k \rangle = \langle G u_j \mid (2H - \lambda_j - \lambda_k) G u_k \rangle . \qquad (1.5)$$

In particular,

$$\langle u_j \mid [G,[H,G]]\, u_j \rangle = 2 \langle G u_j \mid (H - \lambda_j) G u_j \rangle . \qquad (1.6)$$

Commutators and gaps

Since $$[H, G]u_k = (H - \lambda_k)Gu_k,$$

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A trace identity

$$(z-\lambda_j)\,\langle u_j, [G,[H,G]]\,u_j\rangle - 2\|\,[H,G]\,u_j\|^2$$

$$= \sum_k (z-\lambda_k)(\lambda_k-\lambda_j)|\,\langle u_j, G u_k\rangle\,|^2$$

Canonical commutation

Suppose now that H is a Schrödinger operator of standard type, $H = - \nabla^2 + V(x)$, on a Euclidean domain, and that G is a Euclidean coordinate x_k. Then $[H,G] = -2\partial/\partial x_k$, and the second commutator $[G, [H, G]] = 2$.

Physical interpretation: Up to scalar factors, [H,G] is a momentum, and $[G, [H, G]] = 2$ is a form of the Heisenberg commutation relation.

In 1925 Heisenberg used commutation to derive identities to explain the experimentally observed Thomas-Reiche-Kuhn *sum rules*.

Universal Bounds using Commutators

- A "sum rule" identity (Harrell-Stubbe, 1997):

$$1 = \frac{4}{d} \sum_{k:\lambda_k \neq \lambda_j} \frac{|\langle u_k, \mathbf{p} u_j \rangle|^2}{\lambda_k - \lambda_j}$$

Here, H is *any* Schrödinger operator on flat space, **p** is the gradient (times –i by physicist's conventions)

Universal Bounds using Commutators

$$1 = \frac{4}{d} \sum_{k:\lambda_k \neq \lambda_j} \frac{|\langle u_k, \mathbf{p} u_j \rangle|^2}{\lambda_k - \lambda_j}$$

- No sum on j - multiply by $f(\lambda_j)$, sum and symmetrize
- Numerator only kinetic energy - no potential.

Gap Lemma

Lemma 1.1 *Let H be a positive self-adjoint operator with discrete eigenvalues λ_1 and λ_2. Let P denote the orthogonal projection onto u_1, and suppose G is a self-adjoint operator such that the products GP, G^2P, HG^2P, H^2GP, and $GHGP$ are defined. Then the fundamental gap $\Gamma := \Gamma(H)$ satisfies*

$$\Gamma \langle u_1, [G,[H,G]]\, u_1 \rangle \leq 2 \| [H,G] u_1 \|^2. \tag{1.7}$$

The Serret-Frenet equations as commutator relations:

$$[H, X_m] = -\frac{d^2 X_m}{ds^2} - 2\frac{dX_m}{ds}\frac{d}{ds} = -\kappa n_m - 2t_m\frac{d}{ds}, \tag{2.2}$$

$$[X_m, [H, X_m]] = 2t_m^2. \tag{2.3}$$

Proposition 2.1 *Let M be a smooth curve in $\mathbb{R}^\nu, \nu = 2$ or 3. Then for* $H = -\dfrac{d^2}{ds^2} + V(s)$ *and* $\varphi \in W_0^1(M)$,

$$\sum_{m=0}^{d} \| [H, X_m]\, \varphi \|^2 = 4 \int_M \left(\left| \frac{d\varphi}{ds} \right|^2 + \frac{\kappa^2}{4} |\varphi|^2 \right) ds.$$

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Proof. By closure it may be assumed that $\varphi \in C_c^\infty(M)$. Apply (2.2) to φ and square the result, to obtain

$$4 \left(t_m^2 \left(\frac{d\varphi}{ds} \right)^2 + \frac{1}{4} \kappa^2 n_m^2 \varphi^2 + \frac{1}{2} \kappa n_m t_m \varphi \frac{d\varphi}{ds} \right).$$

Sum on m and integrate. QED

Corollary 2.2 *Let* M *be as in* Proposition 2.1 *and suppose that* H *is a Schrödinger Hamiltonian with a bounded measurable potential* $V(s)$*. Then*

$$\Gamma \leq 4 \int_M \left(\left(\frac{du_1}{ds} \right)^2 + \frac{\kappa^2}{4} u_1^2 \right) ds. \tag{2.5}$$

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Furthermore, if H is of the form

$$H_g := -\frac{d^2}{ds^2} + g\kappa^2,$$

then

$$\Gamma \leq \max\left(4, \frac{1}{g}\right) \lambda_1. \tag{2.6}$$

Equivalently, the universal ratio bound

$$\frac{\lambda_2}{\lambda_1} \leq \max\left(5, 1 + \frac{1}{g}\right)$$

holds. This bound is sharp for $0 < g \leq \frac{1}{4}$.

Bound is sharp for the circle:

$$\frac{\lambda_2}{\lambda_1} = \frac{4\pi^2(1+g)}{4\pi^2 g} = 1 + \frac{1}{g}.$$

Gaps bounds and spectral identities for (hyper) surfaces

Let M be a d-dimensional manifold immersed in $\mathbb{R}^{d+1}$.

Theorem 3.1 *Let H be a Schrödinger operator on M with a bounded potential, i.e.,*

$$H = -\Delta + V, \tag{3.1}$$

where V is a bounded, measurable, real-valued function on M. If M has a boundary, Dirichlet conditions are imposed (in the weak sense that H is defined as the Friedrichs extension from $C_c^\infty(M)$). Then

$$\begin{aligned}\Gamma(H) &\le \frac{1}{d}\int_M \left(4\left|\nabla_{||}u_1\right|^2 + h^2 u_1^2\right) dVol \\ &= \frac{4}{d}\left\langle u_1, \left(-\Delta + \frac{h^2}{4}\right) u_1 \right\rangle.\end{aligned} \tag{3.2}$$

Here h is the sum of the principal curvatures.

Corollary 3.2 *Let H be as in (3.1) and define $\delta := \sup_M \left(\frac{h^2}{4} - V\right)$. Then*

$$\Gamma(H) \leq \frac{4}{d}\left(\lambda_1 + \delta\right).$$

A further corollary is an isoperimetric spectral theorem for operators of the form H_g from (1.10):

Corollary 3.3 *Let H_g be defined on M, a d-dimensional manifold smoothly immersed in $\mathbb{R}^{d+1}$. Then the eigenvalues satisfy*

$$\lambda_2 - \lambda_1 \leq \frac{4\sigma\lambda_1}{d}, \tag{3.7}$$

where $\sigma = \max\left(1, \frac{1}{4g}\right).$

Bound is sharp for the sphere:

$$\lambda_1 = gd^2, \quad \lambda_2 = gd^2 + d$$

$$d = \lambda_2 - \lambda_1 \leq \left(\frac{gd^2}{gd}\right) = d.$$

Spinorial Canonical Commutation

$$\mathbf{P} = \sum_{j=1}^{d} \left(\mathbf{t}_j \frac{\partial}{\partial s_j} \pm \frac{1}{2} \kappa_j \mathbf{n} \right) \tag{4.1}$$

and for a dense set of functions φ,

$$\|\mathbf{P}\varphi\|^2 = \left\langle \varphi, H_{1/4}\varphi \right\rangle . \tag{4.2}$$

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and for a dense set of functions φ,

$$\|\mathbf{P}\varphi\|^2 = \langle \varphi, H_{1/4}\varphi \rangle . \tag{4.2}$$

Thus $\mathbf{P}$ plays the rôle of a momentum operator, with which there is a version of canonical commutation (cf. (1.9)) as follows. Defining a variant commutator bracket for operators $L^2(M) \to \mathbb{R}^{d+1} \otimes L^2(M)$ by $[A; B] := A \cdot B - B \cdot A$, a calculation shows that $[\mathbf{P}; X_k \mathbf{e}_k] = \sum_{j=1}^{d} \mathbf{t}_j \cdot \frac{\partial X_k \mathbf{e}_k}{\partial s_j} = \mathbf{1}$ (identity operator), and by averaging on k,

$$\mathbf{1} = \frac{1}{d} [\mathbf{P}; \mathbf{X}] \tag{4.3}$$

which is a coordinate-independent formula.

Sum Rules

Proposition 4.1 *Let H be as in (3.1), with eigenvalues $\{\lambda_k\}$ and normalized eigenfunctions $\{u_k\}$. Then*

$$1 = \frac{4}{d} \sum_{\substack{k \\ \lambda_k \neq \lambda_j}} \frac{|\langle u_k, \mathbf{P} u_j \rangle|^2}{\lambda_k - \lambda_j}. \tag{4.4}$$

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Furthermore, if f is any function summable on the spectrum $\sigma(H)$, then

$$\sum_j^{\infty} f(\lambda_j) = -\frac{2}{d} \sum_{\substack{j,k \\ \lambda_k \neq \lambda_j}} |\langle u_k, \mathbf{P} u_j \rangle|^2 \frac{f(\lambda_j) - f(\lambda_k)}{\lambda_j - \lambda_k}. \tag{4.5}$$

Sharp universal bound for all gaps

Corollary 4.4 b) *For H_q be of the form (1.10) on a smooth, compact submanifold. Then*

$$[\lambda_n, \lambda_{n+1}] \subseteq \left[\left(1 + \frac{2\sigma}{d}\right)\overline{\lambda_n} - \sqrt{D_n}, \left(1 + \frac{2\sigma}{d}\right)\overline{\lambda_n} + \sqrt{D_n}\right],$$

with

$$D_n := \left(\left(1 + \frac{2\sigma}{d}\right)\overline{\lambda_n}\right)^2 - \left(1 + \frac{4\sigma}{d}\right)\overline{\lambda_n^2}.$$

This bound is sharp for every non-zero eigenvalue gap of $H_{\frac{1}{4}}$ on the sphere.

Submanifolds (arbitrary codimension) - with El Soufi & Ilias

Theorem 2.1 *Let $X : M \longrightarrow \mathbb{R}^m$ be an isometric immersion. We denote by h the mean curvature vector field of X (i.e the trace of its second fundamental form). For any bounded potential q on M, the spectrum of $H = -\Delta + q$ (with Dirichlet boundary conditions if $\partial M \neq \emptyset$) must satisfy, $\forall k \geq 1$,*

$$(I)\ n \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq 4 \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) (\lambda_i + \delta_i)$$

$$\delta_i := \int_M \left(\frac{|h|^2}{4} - q \right) u_i^2$$

Submanifolds - Result is optimal

$$n\sum_{i=1}^{k}\left(\lambda_{k+1}^{sphere}-\lambda_i^{sphere}\right)^2=\sum_{i=1}^{k}\left(\lambda_{k+1}^{sphere}-\lambda_i^{sphere}\right)\left(4\lambda_i^{sphere}+n^2\right)$$

Theorem 3.1 *Let $\overline{M}$ be $\mathbb{S}^m$ or $\mathbb{F}P^m$ and let $X : M \longrightarrow \overline{M}$ be an isometric immersion of mean curvature h. For any bounded potential q on M, the spectrum of $H = -\Delta_g + q$ (with Dirichlet boundary conditions if $\partial M \neq \emptyset$) must satisfy, $\forall k \in \mathbb{N}$, $k \geq 1$,*

$$\text{(I)}\ n\sum_{1}^{k}(\lambda_{k+1} - \lambda_i)^2 \leq 4\sum_{i=1}^{k}(\lambda_{k+1} - \lambda_i)\left(\lambda_i + \bar{\delta}_i\right),$$

where $\bar{\delta}_i := \frac{1}{4}\int_M (|h|^2 + c(n) - 4q)u_i^2$,

$$\text{(II)}\ \lambda_{k+1} \leq \left(1 + \frac{2}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{2}{n}\frac{1}{k}\sum_{i=1}^{k}\bar{\delta}_i + \sqrt{\bar{D}_{nk}}$$

where

$$\bar{D}_{nk} := \left(\left(1 + \frac{2}{n}\right)\frac{1}{k}\sum_{1}^{k}\lambda_i + \frac{2}{n}\frac{1}{k}\sum_{i=1}^{k}\bar{\delta}_i\right)^2 - \left(1 + \frac{4}{n}\right)\frac{1}{k}\sum_{1}^{k}\lambda_i^2 - \frac{4}{n}\frac{1}{k}\sum_{i=1}^{k}\lambda_i\bar{\delta}_i \geq 0,$$

A lower bound is also possible along the lines of Theorem 2.1. As in the previous section, the following simplifications follow easily:

Corollary 3.1 *With the notation of Theorem 3.1 one has, $\forall k \geq 1$,*

$$\lambda_{k+1} \leq \left(1 + \frac{4}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i + \frac{4}{n}\,\bar{\delta},$$

where $\bar{\delta} := \frac{1}{4}\sup\left(|h|^2 + c(n) - 4q\right)$.

Riesz means and how to get spectral information from them

These ideas will be illustrated for the Laplacian on a Euclidean domain (joint work with L. Hermi).

Universal bounds of the form λ_k / λ_1

Bounds of this form follow from the bounds on λ_{k+1}, but with bad constants.

Universal bounds of the form λ_k / λ_1

Some previous work:

Universal bounds of the form λ_k / λ_1

Some previous work:

Ashbaugh-Benguria, 1994:

$$\frac{\lambda_{2^m}}{\lambda_1} \leq \left(\frac{j^2_{d/2,1}}{j^2_{d/2-1,1}} \right)^m$$

(Not Weyl type)

Hermi, TAMS to appear:

$$\frac{\lambda_{k+1}}{\lambda_1} \leq 1 + \left(1 + \frac{d}{2}\right)^{2/d} H_d^{2/d}\, k^{2/d},$$

and

$$\frac{\overline{\lambda}_k}{\lambda_1} \leq 1 + \frac{H_d^{2/d}}{1 + \frac{2}{d}}\, k^{2/d},$$

Cheng-Yang, Math. Ann., 2007:

$$\frac{\lambda_{k+1}}{\lambda_1} \leq C_0(d,k)k^{\frac{2}{d}}$$

where in its simplest form, $C_0 = (1 + 4/d)$.

When d=2, the CY bound is more than 4 times the Weyl asymptotics,

$$\frac{4\left(\Gamma(1+\frac{d}{2})\right)^{\frac{4}{d}}}{j^2_{\frac{d}{2}-1,1}}k^{\frac{2}{d}}$$

Ratios of Averages

$$\overline{\lambda_k} := \tfrac{1}{k}\textstyle\sum_{\ell \le k} \lambda_\ell$$

$$\overline{\lambda_j^2} := \frac{1}{j}\sum_{\ell \le j} \lambda_\ell^2$$

Ratios of Averages

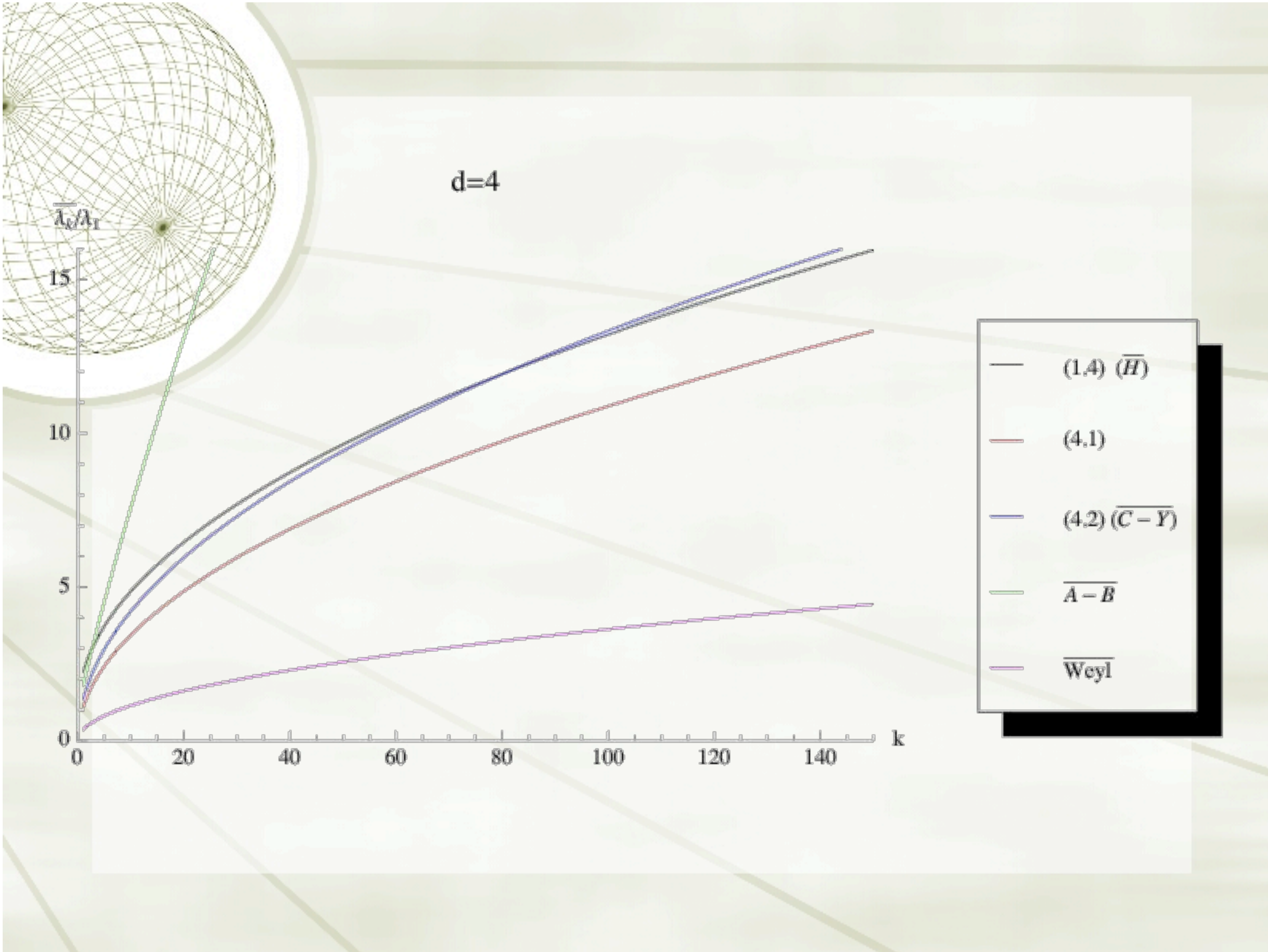
Corollary 3.1 *For $k \geq j \frac{1+\frac{d}{2}}{1+\frac{d}{4}}$, the means of the eigenvalues of the Dirichlet Laplacian satisfy a universal Weyl-type bound,*

$$\overline{\lambda_k}/\overline{\lambda_j} \leq 2 \left(\frac{1+\frac{d}{4}}{1+\frac{d}{2}} \right)^{1+\frac{2}{d}} \left(\frac{k}{j} \right)^{\frac{2}{d}} . \tag{3.4}$$

Ratios of Averages

Corollary 3.2 *For* $k \geq \frac{(d+1)\,(1+\frac{d}{2})}{1+\frac{d}{4}}$,

$$\overline{\lambda_k}/\lambda_1 \leq \frac{d+5}{2^{\frac{2}{d}}} \left(\frac{(d+4)}{(d+1)(d+2)} \right)^{1+\frac{2}{d}} k^{\frac{2}{d}}. \tag{3.5}$$



d=4
$\overline{\lambda_k}/\lambda_1$
0
5
10
15
0
20
40
60
80
100
120
140
k
(1,4) ($\overline{H}$)
(4,1)
(4,2) ($\overline{C-Y}$)
$\overline{A-B}$
$\overline{\text{Weyl}}$

Riesz means

$R_\sigma(z) := \sum (z-\lambda_k)_+^\sigma$ for $\sigma > 0$.

When $\sigma=0$, interpret as limit from above, i.e. the spectral counting function.

Theorem 2.1 *For $0 < \sigma \leq 2$ and $z \geq \lambda_1$,*

$$R_{\sigma-1}(z) \geq \left(1 + \frac{d}{4}\right) \frac{1}{z} R_\sigma(z); \tag{2.2}$$

$$R'_\sigma(z) \geq \left(1 + \frac{d}{4}\right) \frac{\sigma}{z} R_\sigma(z); \tag{2.3}$$

and consequently

$$\frac{R_\sigma(z)}{z^{\sigma + \frac{d\sigma}{4}}}$$

is a nondecreasing function of z.

For $2 \leq \sigma < \infty$ and $z \geq \lambda_1$,

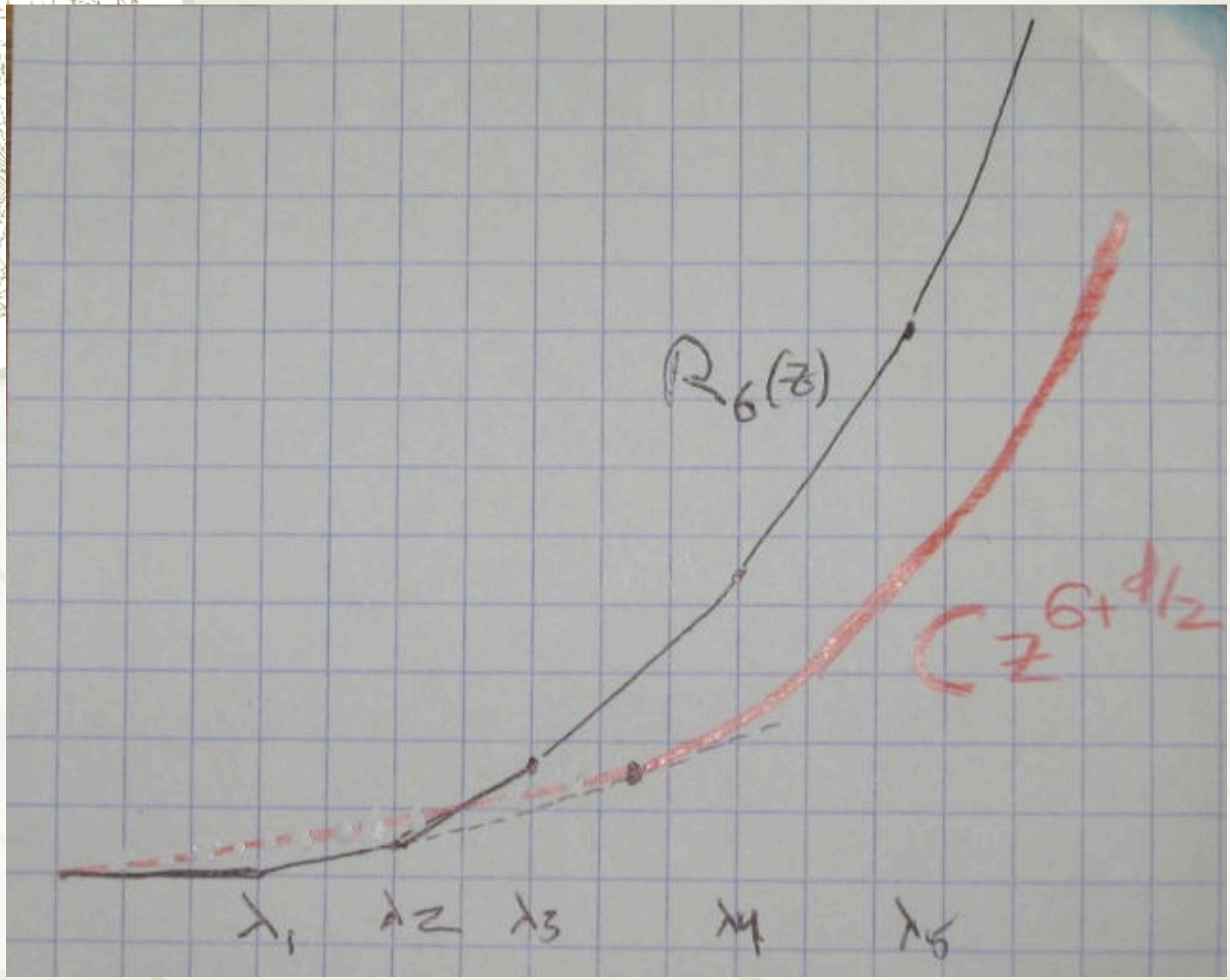
$$R_{\sigma-1}(z) \geq \left(1 + \frac{d}{2\sigma}\right) \frac{1}{z} R_\sigma(z); \tag{2.4}$$

$$R'_\sigma(z) \geq \left(\sigma + \frac{d}{2}\right) \frac{1}{z} R_\sigma(z); \tag{2.5}$$

and consequently

$$\frac{R_\sigma(z)}{z^{\sigma + \frac{d}{2}}}$$

is a nondecreasing function of z.


$R_6(z)$
$Cz^{6+d/2}$
λ_1
λ_2
λ_3
λ_4
λ_5

Idea of proof

$$\sum_{\lambda_j \in J} f(\lambda_j) = -2 \sum_{\substack{\lambda_j, \lambda_m \in J \\ \lambda_j \neq \lambda_m}} \frac{f(\lambda_j) - f(\lambda_m)}{\lambda_j - \lambda_m} T_{\alpha j m} + 4 \sum_{\substack{\lambda_j \in J \\ \lambda_q \in J^c}} \frac{f(\lambda_j)}{\lambda_q - \lambda_j} T_{\alpha j q},$$

Set $f = (z-\lambda_k)_+^{\sigma}$, so the left side becomes R_σ, and notice that the first term on the right is comparable to

$$- \text{cst} \sum (z-\lambda_k)_+^{\sigma-1} \lambda_k = \text{cst} (R_\sigma(z) - z R_{\sigma-1}(z))$$

$$= \text{cst}\, R_\sigma(z) - \text{cst}\, z R_\sigma'(z)$$

Corollary 2.3 *For all $\sigma \geq 2$ and $z \geq \left(1+\frac{2\sigma}{d}\right)\lambda_1$,*

$$\left(\frac{2\sigma}{d}\right)^{\sigma}\lambda_1^{-\frac{d}{2}}\left(\frac{z}{1+\frac{2\sigma}{d}}\right)^{\sigma+\frac{d}{2}} \leq R_\sigma(z) \leq L^{cl}_{\sigma,d}|\Omega| z^{\sigma+\frac{d}{2}}, \tag{2.11}$$

where

$$L^{cl}_{\sigma,d} := \frac{\Gamma(\sigma+1)}{(4\pi)^{\frac{d}{2}}\Gamma\left(\sigma+1+\frac{d}{2}\right)}. \tag{2.12}$$

$$R_1(z) \geq \left(1+\frac{d}{4}\right)\frac{1}{z}R_2(z) \geq \frac{4\ d^{\frac{d}{2}}}{(d+4)^{1+\frac{d}{2}}}\lambda_1^{-\frac{d}{2}} z^{1+\frac{d}{2}}, \tag{2.16}$$

and,

$$\mathcal{N}(z) = R_0(z) \geq \left(1+\frac{d}{4}\right)^2 \frac{1}{z^2}R_2(z) \geq \left(\frac{z}{\left(1+\frac{4}{d}\right)\lambda_1}\right)^{\frac{d}{2}}. \tag{2.17}$$

Riesz means

$R_\sigma(z) := \sum (z-\lambda_k)_+^\sigma$ for $\sigma > 0$.

How is this related to moments of eigenvalues, like

$$\sum \lambda_k^\tau$$

or, equivalently, to averages such as

$$\overline{\lambda_k} := \frac{1}{k} \sum_{\ell \leq k} \lambda_\ell$$

$$\overline{\lambda_j^2} := \frac{1}{j} \sum_{\ell \leq j} \lambda_\ell^2 \qquad ?$$

Legendre transform

Legendre transform

$$\mathcal{L}[f](w) := \sup_z \{wz - f(z)\}$$

$$R_1(z) \geq \frac{4\, d^{\frac{d}{2}}}{(d+4)^{1+\frac{d}{2}}} \lambda_1^{-\frac{d}{2}} z^{1+\frac{d}{2}}$$

becomes ….

Legendre transform

$$(w-[w])\,\lambda_{[w]+1}+[w]\,\overline{\lambda_{[w]}}\le \frac{2}{j^{\frac{2}{d}}}\left(\frac{1+\frac{d}{4}}{1+\frac{d}{2}}\right)^{1+\frac{2}{d}}\overline{\lambda_j}\;w^{1+\frac{2}{d}}$$

Meanwhile, for any w we can always find an integer k such that on the left side of (3.2), $k-1\le w<k$. If $k>j\frac{1+\frac{d}{2}}{1+\frac{d}{4}}$ and if we let w approach k from below, we obtain from (3.2)

$$\lambda_k+(k-1)\overline{\lambda_{k-1}}\le \frac{2}{j^{\frac{2}{d}}}\left(\frac{1+\frac{d}{4}}{1+\frac{d}{2}}\right)^{1+\frac{2}{d}}\overline{\lambda_j}k^{1+\frac{2}{d}}.$$

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